Dissipation due to fermions in inflaton equations of motion

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According to quantum field theory, the inflaton equation of motion does not have the local form that is generally assumed for cosmological purposes. In particular, earlier investigations of the nonequilibrium dynamics of an inflaton that decays into scalar particles suggest that the loss of inflaton energy is not well approximated by the local friction term derived from linear response theory. We extend this analysis to the case of an inflaton that decays into fermions, and reach broadly the same conclusion.

I. INTRODUCTION

In the study of inflationary cosmology, it is often assumed that a classical inflaton field ϕ obeys a local equation of motion of the form

$$\ddot{\phi} + \eta(\phi)\dot{\phi} + V'(\phi) = 0, \qquad (1)$$

where the friction term $\eta(\phi)\dot{\phi}$ accounts for the loss of inflaton energy by radiation of particles. (We omit the usual Hubble damping term $3H\dot{\phi}$, for reasons discussed below.) In particular, "warm inflation" scenarios (see, for example, [1, 2, 3, 4, 5]) envisage a friction coefficient $\eta(\phi)$ large enough for the inflaton motion to be perpetually overdamped, thus avoiding the usual process of reheating.

However, an equation of this form does not arise automatically from quantum field theory. Indeed, if the inflaton couples biquadratically to scalar fields χ, and via Yukawa interactions to spin-$\frac{1}{2}$ fields ψ, its equation of motion is generically of the form

$$\ddot{\phi} + V'(\phi) + g_1\langle\chi^2\rangle\phi + g_2\langle\bar{\psi}\psi\rangle + \cdots = 0 \qquad (2)$$

where the expectation values $\langle\chi^2\rangle$ and $\langle\bar{\psi}\psi\rangle$ are nonlocal functionals of ϕ. That is, they depend on the entire history of ϕ, at all times prior to the time t at which the expectation values are evaluated. The issue we address here is whether these expectation values can be adequately approximated by a local friction term, together with a local correction to the effective potential $V(\phi)$, leading to the equation of motion (1). To focus on this issue, which is substantially independent of the background metric, we simplify matters throughout by studying quantum field theory in Minkowski spacetime, with $H = 0$.

If the motion of ϕ is slow enough, it seems intuitively reasonable that suitable approximations to these expectation values can be obtained by considering the linear response of a state of thermal equilibrium to a time-dependent perturbation, $\dot{\phi}(t)$. Several implementations of this general idea can be found in the literature (see, for example [1, 6, 7, 8, 9, 10]), and

lead to well-defined estimates of the friction coefficient $\eta(\phi)$. However, the *assumption* that this problem can adequately be treated on the basis of linear perturbations about *equilibrium* thermal field theory deserves further investigation. If this assumption is valid, then the results should be reproduced in the slow evolution limit of *nonequilibrium* field theory.

For the case of an inflaton coupled to a scalar field, this does not seem to be true [11, 12, 13]. Formally, one can derive approximate, local nonequilibrium evolution equations, but a time-derivative expansion of the solution to these equations, which is needed to arrive at (1), does not exist. With the further approximation of replacing non-equilibrium self-energies with their equilibrium counterparts, a time-derivative expansion becomes possible, leading to what we call the *adiabatic approximation*, which essentially reproduces the results of linear response theory. However, numerical investigation shows that the motion generated by the nonequilibrium equations is not well approximated by the adiabatic approximation.

These results might well be specific to the particular field theory considered. In this paper, we investigate this to some extent by extending the analysis of [11, 12, 13] to the case of an inflaton that decays into spin-$\frac{1}{2}$ fermions. In principle, the calculations are quite similar, but the technicalities associated with spin-$\frac{1}{2}$ fields are slightly more complicated.

In section II we derive exact evolution equations for the number densities $N_k(t)$, and auxiliary functions $\nu_k(t)$, which together specify the nonequilibrium state of fermions that are free, except for a ϕ-dependent mass arising from a Yukawa coupling to the classical inflaton. These are the fermionic analogues of the exact equations derived for scalar fields by Morikawa and Sasaki [7]. As in the scalar-field case, they do not describe any significant frictional effect, because particles radiated by the inflaton motion can easily be reabsorbed. To achieve a permanent transfer of energy from the inflaton to the system of particles, these particles must interact with each other, and dissipate energy by scattering.

An approximate set of nonequilibrium evolution equations is obtained in section III, by adapting the analysis described by Lawrie and McKernan in [14]. The closed-time-path methods used to develop this approximation are very different from the straightforward (if long-winded) operator methods available for free field theories, and the fact that the equations of section II are recovered in the free-field limit is a crucial check on this method of approximation. We again find that

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the nonequilibrium evolution equations do not admit a solution in the form of a time-derivative expansion (in fact, this seems to be a general, model-independent result), but an adiabatic approximation can be devised, as explained in section III C.

Numerical solutions to the nonequilibrium evolution equations, and to the adiabatic approximation to these equations, are described in section IV. As in the previous case of an inflaton that decays into scalars, we find that these solutions do not agree well. In general, the adiabatic approximation appears to overestimate the damping effect of particle radiation on the inflaton motion; in particular, we find that overdamping predicted by the adiabatic approximation, is not exhibited by the nonequilibrium evolution.

A self-contained summary and discussion of these results is given in section V.

II. FREE-FIELD EVOLUTION EQUATIONS

In this section, we derive a set of exact equations for the time evolution of a system of fermions that are free, apart from a coupling to the classical inflaton; this gives rise to an effective time-dependent mass $m(t) = m_\psi + g\phi(t)$, where m_ψ denotes the bare mass. These equations serve as a valuable check on the validity and interpretation of the approximate evolution equations we obtain in section III for the system of interacting fermions.

We look for a formal solution of the Dirac equation with time dependent mass,

$$[i\gamma^\mu \partial_\mu - m(t)]\psi(x) = 0 \qquad (3)$$

by expanding the field $\psi(x) \equiv \psi(\boldsymbol{x}, t)$ in terms of two positive-energy spinors $u_\pm$ and their charge conjugates $u_\pm^c = i\gamma^2 u_\pm^*$:

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} [A_k(t;\hat{t})u_+(\boldsymbol{k},\hat{t}) + B_k(t;\hat{t})u_-(\boldsymbol{k},\hat{t})$$
$$+ C_k(t;\hat{t})u_+^c(-\boldsymbol{k},\hat{t}) + D_k(t;\hat{t})u_-^c(-\boldsymbol{k},\hat{t})]. \qquad (4)$$

The spinors $u_\pm(\boldsymbol{k},\hat{t})$ are helicity eigenstates, with helicity $\frac{1}{2}h$, $h = \pm 1$, as indicated by the subscript and, at an arbitrary reference time $\hat{t}$, are solutions of the instantaneous Dirac equation

$$\left[\gamma^0 \hat{\Omega}_k + \boldsymbol{\gamma}\cdot\boldsymbol{k} + \hat{m}\right] u_\pm(\boldsymbol{k},\hat{t}) = 0, \qquad (5)$$

where $\hat{m} = m(\hat{t})$ and $\hat{\Omega}_k = \sqrt{k^2 + \hat{m}^2}$, normalized such that $u_h^\dagger(\boldsymbol{k},\hat{t})u_{h'}(\boldsymbol{k},\hat{t}) = \delta_{hh'}$. Dependence on the time t is contained in the functions $A_k(t;\hat{t}), B_k(t;\hat{t}), \ldots$ and these coefficients also depend on $\hat{t}$, in such a way that $\psi(x)$ itself is independent of $\hat{t}$. By substituting the *Ansatz* (4) into the Dirac equation (3), we find equations for the t dependence of $A_k(t;\hat{t}), B_k(t;\hat{t}), \ldots$. The change of variables

$$\begin{array}{ll} A_k = c_k\tilde{A}_k - s_k\tilde{C}_k & B_k = c_k\tilde{B}_k - s_k\tilde{D}_k \\ C_k = s_k\tilde{A}_k + c_k\tilde{C}_k & D_k = -s_k\tilde{B}_k - c_k\tilde{D}_k \end{array} \qquad (6)$$

with

$$c_k = \frac{\hat{\Omega}_k + \hat{m}}{\sqrt{2\hat{\Omega}_k(\hat{\Omega}_k + \hat{m})}} \qquad s_k = \frac{|\boldsymbol{k}|}{\sqrt{2\hat{\Omega}_k(\hat{\Omega}_k + \hat{m})}} \qquad (7)$$

(satisfying $c_k^2 + s_k^2 = 1$) brings these equations into a standard form used in [14], namely

$$i\partial_t \begin{pmatrix} \tilde{A}_k(t;\hat{t}) \\ \tilde{C}_k(t;\hat{t}) \end{pmatrix} = \boldsymbol{T}_k(t) \begin{pmatrix} \tilde{A}_k(t;\hat{t}) \\ \tilde{C}_k(t;\hat{t}) \end{pmatrix}$$
$$i\partial_t \begin{pmatrix} \tilde{B}_k(t;\hat{t}) \\ \tilde{D}_k(t;\hat{t}) \end{pmatrix} = \boldsymbol{T}_k(t) \begin{pmatrix} \tilde{B}_k(t;\hat{t}) \\ \tilde{D}_k(t;\hat{t}) \end{pmatrix}. \qquad (8)$$

The matrix

$$\boldsymbol{T}_k(t) = \begin{pmatrix} m(t) & -k \\ -k & -m(t) \end{pmatrix} \qquad (9)$$

(with $k = |\boldsymbol{k}|$) has eigenvalues $\pm\Omega_k(t)$, where $\Omega_k(t) = \sqrt{m(t)^2 + k^2}$. Correspondingly, two orthogonal solutions to the equation $i\partial_t \boldsymbol{F}(t) = \boldsymbol{T}_k(t)\boldsymbol{F}(t)$ can be written as

$$\boldsymbol{F}_k^{(+)}(t;\hat{t}) \equiv \begin{pmatrix} f_k(t;\hat{t}) \\ g_k(t;\hat{t}) \end{pmatrix} = \mathcal{T} e^{-i\int_{\hat{t}}^t \boldsymbol{T}_k(t')dt'} \begin{pmatrix} c_k \\ -s_k \end{pmatrix} (10)$$

$$\boldsymbol{F}_k^{(-)}(t;\hat{t}) \equiv \begin{pmatrix} -g_k^*(t;\hat{t}) \\ f_k^*(t;\hat{t}) \end{pmatrix} = \mathcal{T} e^{-i\int_{\hat{t}}^t \boldsymbol{T}_k(t')dt'} \begin{pmatrix} s_k \\ c_k \end{pmatrix} (11)$$

where $\mathcal{T}$ denotes time ordering. At the reference time $\hat{t}$, $\boldsymbol{F}_k^{(+)}$ and $\boldsymbol{F}_k^{(-)}$ are positive- and negative-energy solutions respectively (that is, $i\partial_t \boldsymbol{F}_k^{(\pm)}\big|_{t=\hat{t}} = \pm\hat{\Omega}_k \boldsymbol{F}_k^{(\pm)}$).

The general solution for $\tilde{A}_k(t;\hat{t}), \tilde{B}_k(t;\hat{t}), \ldots$ can thus be written as

$$\begin{pmatrix} \tilde{A}(t;\hat{t}) \\ \tilde{C}(t;\hat{t}) \end{pmatrix} = b_+(\boldsymbol{k},\hat{t}) \begin{pmatrix} f_k(t;\hat{t}) \\ g_k(t;\hat{t}) \end{pmatrix} + d_+^\dagger(-\boldsymbol{k},\hat{t}) \begin{pmatrix} -g_k^*(t;\hat{t}) \\ f_k^*(t;\hat{t}) \end{pmatrix}$$
$$\begin{pmatrix} \tilde{B}(t;\hat{t}) \\ \tilde{D}(t;\hat{t}) \end{pmatrix} = b_-(\boldsymbol{k},\hat{t}) \begin{pmatrix} f_k(t;\hat{t}) \\ g_k(t;\hat{t}) \end{pmatrix} - d_-^\dagger(-\boldsymbol{k},\hat{t}) \begin{pmatrix} -g_k^*(t;\hat{t}) \\ f_k^*(t;\hat{t}) \end{pmatrix}$$
$$(12)$$

and the canonical equal-time anticommutator of $\psi(x)$ and $\psi^\dagger(x)$ implies that the coefficients $b_h(\boldsymbol{k},\hat{t})$ and $d_h(\boldsymbol{k},\hat{t})$ have the usual anticommutator algebra

$$\{b_h(\boldsymbol{k},\hat{\boldsymbol{t}}), b_{h'}^\dagger(\boldsymbol{k}',\hat{t})\} = (2\pi)^3 \delta(\boldsymbol{k}-\boldsymbol{k}')\delta_{hh'}, \text{etc}$$

for particle creation and annihilation operators.

To find how these operators depend on $\hat{t}$ we make use of the fact that $\partial_{\hat{t}}\psi(x) = 0$. On substituting (6) and (12) into (4), we find (after some algebra) that

$$\partial_{\hat{t}} b_h(\boldsymbol{k},\hat{t}) = -i\hat{\Omega}_k b_h(\boldsymbol{k},\hat{t}) + h\lambda_k d_h^\dagger(-\boldsymbol{k},\hat{t})$$
$$\partial_{\hat{t}} d_h(\boldsymbol{k},\hat{t}) = -i\hat{\Omega}_k d_h(\boldsymbol{k},\hat{t}) - h\lambda_k b_h^\dagger(-\boldsymbol{k},\hat{t}) \qquad (13)$$

where

$$\lambda_k = \frac{k\partial_{\hat{t}}\hat{\Omega}_k}{2\hat{\Omega}_k\hat{m}} = \frac{k\partial_{\hat{t}}\hat{m}}{2\hat{\Omega}_k^2}. \qquad (14)$$

In a translation- and rotation-invariant state, expectation values of bilinear products of creation and annihilation operators are characterized by functions $N_h(k,\hat{t})$, $\bar{N}_h(k,\hat{t})$ and $\nu_h(k,\hat{t})$, defined by

$$\langle b_h^\dagger(\boldsymbol{k},\hat{t}) b_{h'}(\boldsymbol{k}',\hat{t})\rangle = (2\pi)^3 \delta(\boldsymbol{k}-\boldsymbol{k}')\delta_{hh'} N_h(k,\hat{t})$$

$$\langle d_h^\dagger(\boldsymbol{k},\hat{t}) d_{h'}(\boldsymbol{k}',\hat{t})\rangle = (2\pi)^3 \delta(\boldsymbol{k}-\boldsymbol{k}')\delta_{hh'} \bar{N}_h(k,\hat{t}) \quad (15)$$

$$\langle d_h(-\boldsymbol{k},\hat{t}) b_{h'}(\boldsymbol{k}',\hat{t})\rangle = (2\pi)^3 \delta(\boldsymbol{k}-\boldsymbol{k}')\delta_{hh'} \nu_h(k,\hat{t}).$$

Because the mode functions $\boldsymbol{F}_k^{(+)}(t,\hat{t})$ are positive-frequency only for times t near $\hat{t}$, we can interpret $N_h(k,\hat{t})$ as the number density of particles present *at time* $\hat{t}$ with momentum $\boldsymbol{k}$ and helicity h, and $\bar{N}_h(k,\hat{t})$ as the corresponding density of antiparticles. The functions $\nu_h(k,\hat{t})$ measure the off-diagonality of the density matrix in the basis specified by the modes $\boldsymbol{F}_k^{(\pm)}$, and have no classical analogue.

The expectation value $\langle \bar{\psi}(\boldsymbol{x},t)\psi(\boldsymbol{x},t)\rangle$ is conveniently expressed in terms of the number densities at time t, and we achieve this by setting $\hat{t}=t$. We will restrict attention to states in which $N_+(k,t)$, $N_-(k,t)$, $\bar{N}_+(k,t)$ and $\bar{N}_-(k,t)$ are all equal, say to $N_k(t)$, while $\nu_+(k,t) = -\nu_-(k,t) \equiv \nu_k(t)$. It is easily checked using (13) that these equalities are preserved by the time evolution, and that the real number density $N_k(t)$ and the complex function $\nu_k(t)$ obey the evolution equations

$$\partial_t N_k(t) = \frac{k\dot{m}(t)}{\Omega_k^2(t)} \nu_k^R(t) \quad (16)$$

$$\partial_t \nu_k(t) = -2i\Omega_k(t)\nu_k(t) - \frac{k\dot{m}(t)}{2\Omega_k^2(t)}\left[2N_k(t)-1\right], \quad (17)$$

where ν^R is the real part of ν. The expectation value of interest is given in terms of these functions by

$$\langle \bar{\psi}(\boldsymbol{x},t)\psi(\boldsymbol{x},t)\rangle = 4\int \frac{d^3k}{(2\pi)^3 2\Omega_k}\left[m(2N_k-1)+2k\nu_k^R\right], \quad (18)$$

with all the quantities m, Ω_k, N_k, ν_k evaluated at time t. The evolution equations (16) and (17) are the fermionic analogues of those derived by Morikawa and Sasaki [7] for the case of the inflaton field coupled to a scalar field. The right-hand side of (16) is a particle 'creation' term, proportional to $d\phi/dt$, but it can be either positive or negative as ν_k^R oscillates, and a numerical solution shows that there is no permanent transfer of energy from the inflaton to the system of fermions. To see a frictional effect, it is necessary to introduce interactions between the fermions so that, roughly speaking, the created particles lose coherence with the inflaton, and a permanent dissipation of energy results.

III. EVOLUTION EQUATIONS FOR AN INTERACTING SYSTEM

A. Derivation

An approximation scheme for studying the non-equilibrium evolution of a system of interacting fermions was developed by Lawrie and McKernan in [14]. We follow their derivation closely, except that we introduce, as in section II, an evolving reference time $\hat{t}$, which allows us to make explicit the effect of particle creation.

Within the closed-time-path (CTP) formalism (see, e.g. [15, 16] and references therein), we seek an approximation to the full two-point function

$$\mathcal{S}_{\alpha\beta}^{(ab)}(t,\boldsymbol{x};t',\boldsymbol{x}') =$$
$$\begin{pmatrix} \langle \mathcal{T}[\psi_\alpha(t,\boldsymbol{x})\bar{\psi}_\beta(t',\boldsymbol{x}')]\rangle & -\langle \bar{\psi}_\beta(t',\boldsymbol{x}')\psi_\alpha(t,\boldsymbol{x})\rangle \\ \langle \psi_\alpha(t,\boldsymbol{x})\bar{\psi}_\beta(t',\boldsymbol{x}')\rangle & \langle \bar{\mathcal{T}}[\psi_\alpha(t,\boldsymbol{x})\bar{\psi}_\beta(t',\boldsymbol{x}'))]\rangle \end{pmatrix} \quad (19)$$

from which the expectation value $\langle \bar{\psi}(\boldsymbol{x},t)\psi(\boldsymbol{x},t)\rangle$ can be obtained as a special case. (Here, α, β are Dirac indices, while $a, b = 1,2$ distinguish the indicated operator orderings.) It is shown in [14] that this can be expressed (after a spatial Fourier transform) in terms of a single 4×4 matrix $\mathcal{H}(t,t';\boldsymbol{k})$ as

$$\mathcal{S}^{(ab)}(t,t';\boldsymbol{k}) = \begin{pmatrix} \mathcal{H}(t,t';\boldsymbol{k}) & \tilde{\bar{\mathcal{H}}}(t,t';\boldsymbol{k}) \\ \mathcal{H}(t,t';\boldsymbol{k}) & \bar{\mathcal{H}}(t,t';\boldsymbol{k}) \end{pmatrix} \theta(t-t')$$
$$+ \begin{pmatrix} \tilde{\mathcal{H}}(t,t';\boldsymbol{k}) & \tilde{\mathcal{H}}(t,t';\boldsymbol{k}) \\ \bar{\mathcal{H}}(t,t';\boldsymbol{k}) & \bar{\mathcal{H}}(t,t';\boldsymbol{k}) \end{pmatrix} \theta(t'-t) \quad (20)$$

For any matrix M, we define $\bar{M} = \gamma^0 M^\dagger \gamma^0$ and $\tilde{M} = [C^{-1}\gamma^0 M\gamma^0 C]^T$ where C is the charge conjugation matrix and ^{T} indicates the transpose.

As explained in detail in [14] (and in [12, 17] for the case of ϕ coupling to a scalar field), we construct a partially-resummed perturbation theory as follows. Given a CTP action $I = I^{(2)} + I^{(>2)}$, where $I^{(2)}$ is quadratic in the fields and $I^{(>2)}$ is of higher order, we take its lowest-order part to be

$$I_0(\psi) = I^{(2)}(\psi) + \int d^4x\, \bar{\psi}_a \mathcal{M}_{ab}\psi_b \equiv \int d^4x\, \bar{\psi}_a \mathcal{D}_{ab}\psi_b. \quad (21)$$

where $\mathcal{M}$ is a counterterm to be determined, and treat the remainder, $I_{\rm int} = I^{(>2)} - \int d^4x\, \bar{\psi}_a\mathcal{M}_{ab}\psi_b$ as a perturbation. (For $a = 1,2$, ψ_a are the independent path-integration variables that inhabit the real-time branches of the closed time path; we have indicated explicitly only the contribution of one spin-$\frac{1}{2}$ field.) The lowest order propagator $S(t,t')$ (we suppress the momentum argument $\boldsymbol{k}$) obeys the equation

$$\mathcal{D}_{ac}(t,\partial_t)S^{(cb)}(t,t') = S^{(ac)}(t,t')\mathcal{D}_{cb}(t',-\overleftarrow{\partial}_{t'})$$
$$= i\delta_{ab}\delta(t-t') \quad (22)$$

where the differential operator $\mathcal{D}$, defined by (21), is

$$\mathcal{D}(t,\partial_t) = \begin{pmatrix} \mathcal{D}_0(t,\partial_t) & 0 \\ 0 & -\mathcal{D}_0(t,\partial_t) \end{pmatrix} + \mathcal{M}(t), \quad (23)$$

and $\mathcal{D}_0 = i\gamma^0\partial_t - \boldsymbol{\gamma}\cdot\boldsymbol{k} - m(t)$ is the usual Dirac operator. Given suitable constraints on the form of the counterterm $\mathcal{M}$, the solution for $S^{(ab)}(t,t')$ can be written in the form (20) in terms of a function $H(t,t')$ which is the lowest-order approximation to $\mathcal{H}(t,t')$.

In this way, we obtain a reorganized perturbation theory, in which the vertices implied by $I^{(>2)}$ are supplemented by the counterterm $\mathcal{M}$. In particular, self-energies have the form

$$\Sigma_{ab}(t,t') = \mathcal{M}_{ab}(t)\delta(t-t') + \Sigma_{ab}^{\text{loop}}(t,t') \qquad (24)$$

and we can optimize the propagators $S^{(ab)}$ as an approximation to the full 2-point functions $\mathcal{S}^{(ab)}$ by choosing $\mathcal{M}_{ab}$ to cancel a local part of the loop diagrams in $\Sigma_{ab}^{\text{loop}}$.

The solution of (22) is explained in detail in [14]. The matrix $H(t,t')$ from which $S^{(ab)}(t,t')$ is built can be expressed in terms of the γ matrices, the mode functions $\boldsymbol{F}_k^{(\pm)}(t,\hat{t})$ introduced in section II, and two auxiliary functions $N_k(t,\hat{t})$ and $\Delta_k(t,\hat{t})$ which, in the free-field limit, reduce to $N_k(\hat{t})$ and $-\nu_k(\hat{t})$ respectively. These functions satisfy differential equations in the variable t, which arise from (22). In [14], the reference time $\hat{t}$ was taken to be fixed at at the instant when an initial state is specified, but here we find it convenient to allow $\hat{t}$ to vary, and eventually, as in section II, to set $\hat{t} = t$. The dependence of $N_k(t,\hat{t})$ and $\Delta_k(t,\hat{t})$ on $\hat{t}$ can be found from the fact that $S^{(ab)}(t,t')$ must be independent of $\hat{t}$.

We now define

$$N_k(t) = N_k(t,t) \qquad \text{and} \qquad \nu_k(t) = -\Delta_k(t,t). \qquad (25)$$

Evidently, we have

$$\partial_t N_k(t) = \left[\partial_t N_k(t,\hat{t}) + \partial_{\hat{t}} N_k(t,\hat{t})\right]_{\hat{t}=t} \qquad (26)$$

and similarly for $\nu_k(t)$. Applying the results of [14], we find that these two equations have the form

$$\partial_t N_k(t) = -\Gamma_k(t)\left[2N_k(t)-1\right] + \alpha_k(t) + \frac{k\dot{m}(t)}{\Omega_k(t)^2}\nu_k^R(t) \qquad (27)$$

$$\partial_t \nu_k(t) = -2[i\Omega_k(t) + \Gamma_k(t)]\nu_k(t) - \frac{k\dot{m}(t)}{2\Omega_k(t)^2}\left[2N_k(t)-1\right]. \qquad (28)$$

The functions $\Gamma_k(t)$, which can be interpreted as a quasiparticle decay width, and $\alpha_k(t)$ appear in the counterterm $\mathcal{M}$ and, according to the strategy outlined above, are to be determined from the self-energies $\Sigma_{ab}^{\text{loop}}$.

A concrete realization of this strategy is described in [14] for the case that the fermions interact through a Yukawa coupling to a scalar field $\chi(\boldsymbol{x},t)$ of mass M,

$$\mathcal{L}_{\psi\chi} = -g_\chi \bar{\psi}(\boldsymbol{x},t)\psi(\boldsymbol{x},t)\chi(\boldsymbol{x},t), \qquad (29)$$

and we consider the same model here, assuming for simplicity that χ does not couple to the inflaton, so M is constant. Evaluating the relevant self-energies to 1-loop order gives

$$\Gamma_k(t) = \frac{g_\chi^2}{64\pi^2}(M^2-4m^2)\int d^3k' \frac{\delta(\omega_p - \Omega_k - \Omega_{k'})}{\Omega_k \Omega_{k'}\omega_p} \times [n_p + N_{k'}] \qquad (30)$$

$$\alpha_k(t) = \frac{g_\chi^2}{64\pi^2}(M^2-4m^2)\int d^3k' \frac{\delta(\omega_p - \Omega_k - \Omega_{k'})}{\Omega_k \Omega_{k'}\omega_p} \times [n_p(1-N_{k'}) - (1+n_p)N_{k'}]. \qquad (31)$$

Here, m stands for the fermion mass $m(t)$, $n_p(t)$ is (approximately–see below) the number density of scalars, $\boldsymbol{p} = \boldsymbol{k} + \boldsymbol{k}'$ is the momentum of a scalar produced, say, by the collision of two fermions of momenta $\boldsymbol{k}$ and $\boldsymbol{k}'$, and $\omega_p = \sqrt{|\boldsymbol{p}|^2 + M^2}$. In this approximation, the first two terms in equation (27) combine to give

$$\begin{aligned} S_k^\psi(t) &= -\Gamma_k(t)\left[2N_k(t)-1\right] + \alpha_k(t) \\ &= \frac{g_\chi^2}{32\pi^2}(M^2-4m^2)\int d^3k' \frac{\delta(\omega_p - \Omega_k - \Omega_{k'})}{\Omega_k \Omega_{k'}\omega_p} \\ &\times [n_p(1-N_k)(1-N_{k'}) - (1+n_p)N_k N_{k'}] \end{aligned} \qquad (32)$$

which we recognize as the Boltzmann scattering integral corresponding to the decay of a scalar into two fermions and the inverse production process. The evolution of $n_p(t)$ is described by a similar Boltzmann-like equation

$$\begin{aligned} \partial_t n_p &= S_p^\chi(t) \\ &= -\frac{g_\chi^2}{16\pi^2}(M^2-4m^2)\int d^3k' \frac{\delta(\omega_p - \Omega_k - \Omega_{k'})}{\Omega_k \Omega_{k'}\omega_p} \\ &\times [n_p(1-N_k)(1-N_{k'}) - (1+n_p)N_k N_{k'}], \end{aligned} \qquad (33)$$

where now $\boldsymbol{k} = \boldsymbol{p} - \boldsymbol{k}'$. This contains no particle creation term (analogous to the last term of (27)) owing to our assumption that $\chi(\boldsymbol{x},t)$ does not couple to the inflaton.

At this point, we have a closed system of evolution equations, consisting of (27), (28), (33) and the inflaton equation of motion, which we now take to have the form

$$\ddot{\phi} + m_\phi^2 \phi + g\langle\bar{\psi}\psi\rangle_{\text{trunc}} = 0. \qquad (34)$$

Here, we have approximated $\langle\bar{\psi}(\boldsymbol{x},t)\psi(\boldsymbol{x},t)\rangle$ by its lowest-order contribution, $-(2\pi)^{-3}\int d^3k\,\text{Tr}H(t,t',\boldsymbol{k})$. It consists of a local function of $\phi(t)$, which contributes to the effective potential, together with the term

$$\langle\bar{\psi}\psi\rangle_{\text{trunc}} = 4\int \frac{d^3k}{(2\pi)^3\Omega_k(t)}\left[m(t)N_k(t) + k\nu_k^R(t)\right]. \qquad (35)$$

Since we are concerned with the frictional effects arising from $\langle\bar{\psi}\psi\rangle_{\text{trunc}}$, we simplify matters by replacing the whole effective potential with $\frac{1}{2}m_\phi^2\phi^2$. For later use, we observe that the total energy given by

$$E = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m_\phi^2\phi^2 + \int \frac{dk k^2}{2\pi^2}\left[\omega_k n_k + 4\Omega_k N_k\right] \qquad (36)$$

is exactly conserved by this system of evolution equations.

In the interest of accuracy, two remarks are in order. First, the representation (20) of the full 2-point functions is valid for a CP-invariant theory. In practice, this means only that CP-breaking interactions will not be resummed by the counterterm $\mathcal{M}$. Second, the quantities denoted in this section by $N_k(t)$, $\nu_k(t)$ and $n_p(t)$ are functions that arise in the process of solving the appropriate differential equations for spinor and scalar propagators. Only in the free-field limit can they be unambiguously identified in terms of expectation values of creation and annihilation operators as in (15).

B. Renormalization

The integral (35) that appears in the non-equilibrium equation of motion (34) has a divergent contribution from the term ν_k^R in the integrand. To isolate this divergence, consider the evolution equation (28) in the limit that k is large. Assuming that the thermal terms proportional to N_k and Γ_k can be neglected (which is to be expected by analogy with equilibrium field theory, see e.g. section 3.5 of Ref [18], but hard to prove here) we find

$$\partial_t \nu_k(t) \approx -2ik\nu_k(t) + \frac{g}{2k}\dot{\phi}. \tag{37}$$

The solution can be written as

$$\nu_k(t) = e^{-2ikt}\left[\nu_k(0) + \frac{g}{2k}\int_0^t e^{2ikt'}\dot{\phi}(t')dt'\right]. \tag{38}$$

and on repeatedly integrating by parts we find

$$\begin{aligned}\nu_k(t) &= e^{-2ikt}\left[\nu_k(0) + \frac{ig}{4k^2}\dot{\phi}(0) - \frac{g}{8k^3}\ddot{\phi}(0)\right]\\ &\quad - \frac{ig}{4k^2}\dot{\phi}(t) + \frac{g}{8k^3}\ddot{\phi}(t) + \mathcal{O}(k^{-4}).\end{aligned} \tag{39}$$

Now, contributions to $\nu_k^R(t)$ that vanish no faster than k^{-3} as $k \to \infty$ will yield a divergent integral in (35). To avoid this, we firstly choose initial conditions which have the form

$$\nu_k(0) = -\frac{ig}{4k^2}\dot{\phi}(0) + \frac{g}{8k^3}\ddot{\phi}(0) + \mathcal{O}(k^{-4}) \tag{40}$$

for large k, so that the square bracket in (39) vanishes. We are then left with the term $(g/8k^3)\ddot{\phi}(t)$ which can be canceled by a wavefunction renormalization $\phi \to Z^{1/2}\phi$ in the equation of motion (34). This is, in fact, the same wavefunction renormalization that would be required in a zero-temperature quantum field theory with the Yukawa coupling (29). With a suitable choice for Z, the renormalized integral is

$$\begin{aligned}\langle\bar{\psi}\psi\rangle_{\text{trunc}}^{\text{ren}} &= 4\int\frac{dkk^2}{2\pi^2}\left[\frac{m(t)N_k(t) + k\nu_k^R(t)}{\Omega_k(t)}\right.\\ &\quad \left. - \frac{gk}{8(k^2+m_0^2)^2}\ddot{\phi}\right],\end{aligned} \tag{41}$$

where m_0 is a constant. The corresponding renormalized expression for the conserved total energy is

$$\begin{aligned}E^{\text{ren}} &= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m_\phi^2\phi^2 + \int\frac{dkk^2}{2\pi^2}\left[\omega_k n_k + 4\Omega_k N_k\right.\\ &\quad \left. - \frac{g^2 k\dot{\phi}^2}{16(k^2+m_0^2)^2}\right].\end{aligned} \tag{42}$$

Note that, since ν_k^R has a contribution of order k^{-3}, the spectrum of particles created by the last term in (27) has a corresponding contribution of order k^{-4}, and so therefore does N_k. This gives a convergent integral in (41), but a divergent contribution to E^{ren}, which is cancelled by the last term in (42).

C. Adiabatic Approximation

With the approximations developed above, the functions $N_k(t)$ and $\nu_k(t)$ that appear in $\langle\bar{\psi}\psi\rangle_{\text{trunc}}$ obey local evolution equations (27) and (28), but the solution to these equations is non-local, i.e. $\langle\bar{\psi}\psi\rangle_{\text{trunc}}$ evaluated at time t depends on $\phi(t')$ at all times t' prior to t. One can attempt to derive a local approximation of the form

$$g\langle\bar{\psi}\psi\rangle_{\text{trunc}} \approx \Delta V'(\phi) + \eta(\phi)\dot{\phi}, \tag{43}$$

which depends only on the values of ϕ and $\dot{\phi}$ at time t, by means of a time-derivative expansion, but we show below that, as in the case of an inflaton that decays into scalar particles [11], this does not work.

To facilitate a time-derivative expansion, we introduce a formal expansion parameter ϵ multiplying time derivatives

$$\epsilon\partial_t N_k(t) = S_k^\psi + \epsilon\frac{gk\partial_t\phi(t)}{\Omega_k^2(t)}\nu_k^R(t) \tag{44}$$

$$\epsilon\partial_t n_p(t) = S_p^\chi \tag{45}$$

$$\begin{aligned}\epsilon\partial_t\nu_k(t) &= -2[i\Omega_k(t) + \Gamma_k(t)]\nu_k(t)\\ &\quad - \epsilon\frac{gk\partial_t\phi(t)}{2\Omega_k^2(t)}[2N_k(t) - 1],\end{aligned} \tag{46}$$

where we have also used $\dot{m}(t) = g\partial_t\phi(t)$. We now expand $N_k(t)$, $n_p(t)$ and $\nu_k(t)$ in powers of ϵ around the equilibrium distributions $N_k^{\text{eq}}(t)$, $n_p^{\text{eq}}(t)$ and $\nu_k^{\text{eq}}(t)$

$$N_k(t) = N_k^{\text{eq}}(t) + \epsilon\delta N_k(t) + O(\epsilon^2) \tag{47}$$

$$n_p(t) = n_p^{\text{eq}}(t) + \epsilon\delta n_p(t) + O(\epsilon^2) \tag{48}$$

$$\nu_k(t) = \nu_k^{\text{eq}}(t) + \epsilon\delta\nu_k(t) + O(\epsilon^2). \tag{49}$$

Substituting this expansion into equations (44)-(46) we find to leading order that $\nu_k^{\text{eq}} = 0$, while

$$N_k^{\text{eq}}(t) = \left[e^{\beta\Omega_k(t)} + 1\right]^{-1}, \quad n_p^{\text{eq}} = \left[e^{\beta\omega_p} - 1\right]^{-1} \tag{50}$$

for some inverse temperature β are the usual Fermi-Dirac and Bose-Einstein distributions that make the Boltzmann scattering integrals vanish. At order ϵ, δN_k and δn_p are solutions of the integral equations

$$\partial_t N_k^{\text{eq}} = \int dk'\left.\frac{\delta S_k^\psi}{\delta N_{k'}}\right|_{\text{eq}}\delta N_{k'} + \int dp'\left.\frac{\delta S_k^\psi}{\delta n_{p'}}\right|_{\text{eq}}\delta n_{p'}$$

$$\partial_t n_p^{\text{eq}} = \int dk'\left.\frac{\delta S_p^\chi}{\delta N_{k'}}\right|_{\text{eq}}\delta N_{k'} + \int dp'\left.\frac{\delta S_p^\chi}{\delta n_{p'}}\right|_{\text{eq}}\delta n_{p'} \tag{51}$$

where $|_{\text{eq}}$ means that we set $N = N^{\text{eq}}$ and $n = n^{\text{eq}}$ after differentiation. However, these equations are not self-consistent and therefore have no solution. It is easily seen from (32) and (33) that

$$S \equiv 2\int d^3k\, S_k^\psi + \int d^3p\, S_p^\chi = 0. \tag{52}$$

This reflects the fact that if $N = \int d^3k \, N_k$ is the total number density of fermions of each helicity (and also the number of antifermions of each helicity) and $n = \int d^3p \, n_p$ is the total number density of scalars, then the quantity $2N + n$ is conserved by the processes $\chi \leftrightarrow \psi\bar{\psi}$. (For example, one fermion of a given helicity is produced, on average, in every two χ decays.) In the presence of particle production, this conservation law no longer holds, but the scattering integrals still obey the sum rule (52). The integral equations (51) therefore entail

$$2\partial_t \int d^3k \, N_k^{\text{eq}} + \partial_t \int d^3p \, n_p^{\text{eq}}$$
$$= \int dk' \frac{\delta S}{\delta N_{k'}} \delta N_{k'} + \int dp' \frac{\delta S}{\delta n_{p'}} \delta n_{p'}$$
$$= 0, \tag{53}$$

but this is not true for the equilibrium distributions (50). Thus, although the evolution equations themselves are consistent (albeit approximate), they do not admit a solution in the form of a time-derivative expansion. In principle, therefore, we cannot approximate $\langle \bar{\psi}\psi \rangle_{\text{trunc}}$ in the form (43) or the inflaton equation of motion in the local form (1) by means of a time-derivative expansion.

A local approximation to the equation of motion *can* however be obtained by resorting to a further approximation, which consists in replacing $N_k(t)$ and $n_p(t)$ with their equilibrium values (50) for the purpose of evaluating the functions Γ_k, which we then denote by Γ_k^{eq}, and α_k. When this is done, the $O(\epsilon)$ equations for δN_k and $\delta\nu_k$ are

$$\partial_t N_k^{\text{eq}} = -2\Gamma_k^{\text{eq}} \delta N_k \tag{54}$$

$$0 = -2[i\Omega_k + \Gamma_k^{\text{eq}}]\delta\nu_k - \frac{gk\dot{\phi}}{2\Omega_k^2} \times [2N_k^{\text{eq}} - 1] . \tag{55}$$

On rearranging these, we find that the inflaton equation of motion can be written as

$$\ddot{\phi} + m_\phi^2\phi + \Delta V'(\phi) + \eta(\phi)\dot{\phi} \approx 0, \tag{56}$$

with

$$\Delta V'(\phi) = \int \frac{dk \, k^2}{2\pi^2 \Omega_k} \left[m(t)N_k^{\text{eq}}(t) + k\nu_k^{R,\text{eq}}(t) \right]$$
$$= \int \frac{dk \, k^2}{2\pi^2 \Omega_k(t)} \left(\frac{m(t)}{e^{\beta\Omega_k(t)} + 1} \right) \tag{57}$$

and

$$\eta(\phi) = \dot{\phi}^{-1} \int \frac{dk \, k^2}{2\pi^2 \Omega_k} \left[m(t)\delta N_k(t) + k\delta\nu_k^R(t) \right]$$
$$= \frac{g}{4\pi^2} \int \frac{dk \, k^2}{\Omega_k(t)^2} \left\{ \frac{\beta m(t)^2 N_k^{\text{eq}}(t)(1 - N_k^{\text{eq}}(t))}{\Gamma_k^{\text{eq}}(t)} \right.$$
$$\left. + \frac{\Gamma_k^{\text{eq}}(t)}{[\Omega_k^2(t) + \Gamma_k^{\text{eq}2}(t)]} \frac{k^2}{2\Omega_k(t)} [1 - 2N_k^{\text{eq}}(t)] \right\} \tag{58}$$

In the zero temperature limit ($N_k^{\text{eq}} = 0$) and taking $\Gamma_k << \Omega_k$ this expression agrees with that found in [10] (apart from a factor of 2 which we have not been able to account for). We

refer to the approximation represented by (56)-(58), together with a suitable prescription for determining the temperature β^{-1}, as the *adiabatic approximation*.

To estimate time evolution over any extended period of time within the adiabatic approximation, we must allow β to change in an appropriate manner. We obtain a suitable prescription by requiring that the energy (36) be conserved by the adiabatic evolution when n_k and N_k have their equilibrium values.

IV. NUMERICAL INVESTIGATION

The approximate nonequilibrium evolution equations developed in section III A lead to the local equation of motion (56) only when we make the extra approximation of replacing self-energies (from which the functions α_k and Γ_k are derived) with their values in a state of thermal equilibrium, together with the time-derivative expansion which then becomes possible. On the face of it, one might expect these further approximations to be fairly harmless, at least for a system that does not evolve too fast, and does not stray too far from thermal equilibrium. However, the numerical calculations described in [13] for the case of an inflaton coupled to scalar particles showed that the nonequilibrium evolution governed by a set of equations analogous to those of section III A is not well represented by the corresponding adiabatic approximation, even under circumstances when that approximation might seem to be reasonably well justified.

We therefore discuss in this section a similar numerical comparison of the time evolution generated by the nonequilibrium equations described in sections III A and III B, with that generated by the adiabatic approximation of section III C. To discretize the nonequilibrium time evolution, we use a semi-implicit method defined schematically by

$$\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \left[\frac{d\boldsymbol{x}}{dt}\bigg|_n + \frac{d\boldsymbol{x}}{dt}\bigg|_{n+1} \right] \frac{\delta t}{2}, \tag{59}$$

where $\boldsymbol{x}$ represents the whole collection of variables ϕ, $\dot{\phi}$, N_k, n_k, and ν_k and n labels time steps. In the absence of the $\bar{\psi}\psi\chi$ interaction, it is straightforward to show that this method is stable for our problem. Crucially, it is also simple enough that we can implement a discretized version of the renormalization scheme of section III B that is exact up to rounding errors. When interactions are included, stability of the resulting set of integro-differential equations is very hard to analyse. We do in fact see evidence of instability when the calculations are pursued for sufficiently long times, but the results presented below are for periods of time for which the effects of this instability appear to be negligible. Moreover, although the renormalized total energy (42) is exactly conserved in the continuum limit, there is no local discretized version of this expression that is exactly conserved by the discretized evolution equations. As a rough check on our computations, we have evaluated a discretized version of (42), but we do not expect it to be exactly conserved. In particular, divergent contributions to the integral, which cancel by construction in the

continuum limit, are not guaranteed to cancel when time evolution is discretized. In practice, all the initial conditions we have considered lead to an oscillatory solution for $\phi(t)$; we find that energy acquires an oscillatory component (with amplitude of order 5% of the total), consistent with incomplete cancellation of the divergent integral proportional to $\dot\phi^2$, together with a drift amounting to a few percent over the time intervals of interest.

The adiabatic approximation is not affected by these issues. Here, no renormalization is required; semi-implicit discretization of the one remaining differential equation, the equation of motion (56), leads to a stable problem; and exact conservation of energy is imposed as the criterion for determining β. Intuitively, this approximation ought to be good when $\phi(t)$ is slowly varying and thermalization of the created particles is efficient. A rough criterion for efficient thermalization is

$$\tau \, |dm/dt| \ll m \tag{60}$$

or,

$$\sigma \equiv \left| g\dot\phi/m\Gamma_{k=0} \right| \ll 1 \tag{61}$$

where τ is a relaxation time, which we take to be $\tau = 1/\Gamma_{k=0}$.

To obtain slow variation in $\phi(t)$, we would ideally like to identify a set of parameters for which the motion is overdamped. Within the adiabatic approximation, one way of achieving this is to make the first term in the friction coefficient (58) large, by making Γ_k relatively small, but this tends to make thermalization slow, at least according to the criterion (61). The parameter space of couplings, masses and initial conditions is, even for the simple model considered here, too large to permit a systematic exploration, but we have not succeeded in locating a set of parameter values that leads both to overdamped motion and to a very small value of σ. (This was also true of the investigation reported in [13] for an inflaton decaying into scalars; in the context of warm inflation, it seems to be generally hard to devise simple field theory models that exhibit efficient thermalization as well as other desirable properties [4, 9, 19].)

Nevertheless, we can find a situation in which the adiabatic approximation predicts overdamped motion, and σ is not too large. Figure 1 shows the evolution of $\phi(t)$ calculated from the nonequilibrium evolution equations and from the adiabatic approximation, with the parameter values $g = g_\chi = 1$, $m_\phi = m_\chi = 3$, $m_\psi = 0.1$, and initial conditions $\phi(0) = 1$, $\dot\phi(0) = 0$, $\beta = 1$. The solid curve is the motion generated by the adiabatic approximation, which is indeed overdamped; we see, however, that the motion generated by the non-equilibrium equations without the adiabatic approximation (dashed curve) although it exhibits some damping is far from overdamped.

As in [13], this striking discrepancy does not seem to be accounted for by inefficient thermalization. In figure 2 we show the early stages of the motion, along with the parameter σ as estimated within the adiabatic approximation. During this period of time, σ remains below 1. Thus, within the adiabatic approximation, one might well conclude that this approximation is reasonably self-consistent. While thermalization is not
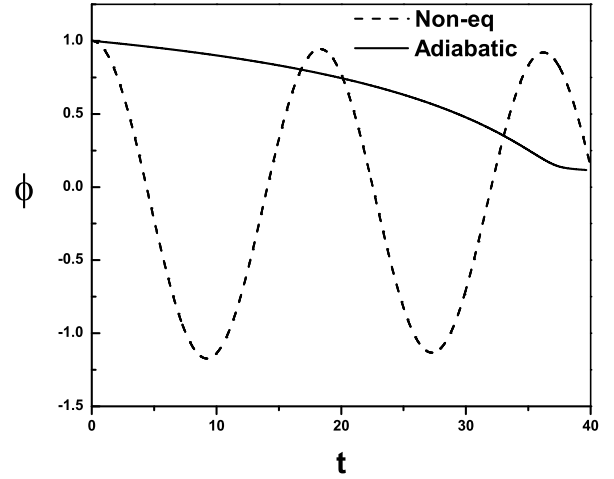

FIG. 1: Evolution of ϕ in both the adiabatic and non-equilibrium solutions. The initial conditions predict overdamping for the adiabatic solution.
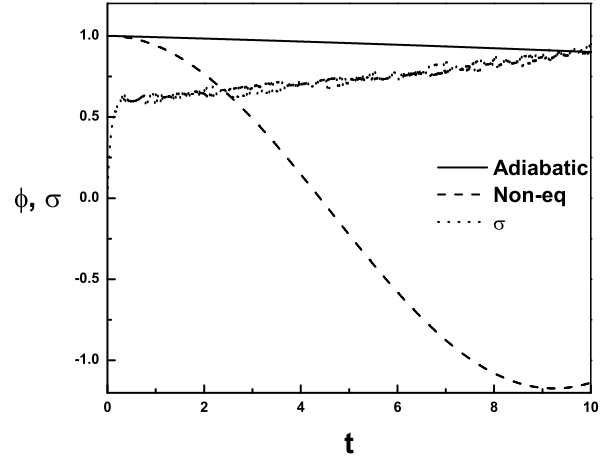

FIG. 2: Evolution of ϕ and σ for early times. The thermalization is relatively efficient in this period and the two solutions are still very different

efficient enough to guarantee good accuracy, one would probably not anticipate the gross discrepancy between the two approximations for $\phi(t)$. Moreover, we compare in figure 3 the function Γ_k calculated at $t \approx 3$ from the nonequilibrium number densities with that calculated in the adiabatic approximation from an exactly thermal distribution. While some difference is visible, it seems clear that the small quantitative difference cannot in itself account for difference between the overdamped and oscillatory motions apparent in figure 2.

In fact, the difference between these motions can be understood in the way suggested in [13]. At $t = 0$, we chose an initial state of exact thermal equilibrium. Denote by $A(\phi)$ the integral (35) evaluated in this state, and let $U(\phi) = m_\phi^2\phi + A(\phi)$. At early times, when the value of ϕ has not changed by much, the nonequilibrium equation of motion (34) is given approximately by

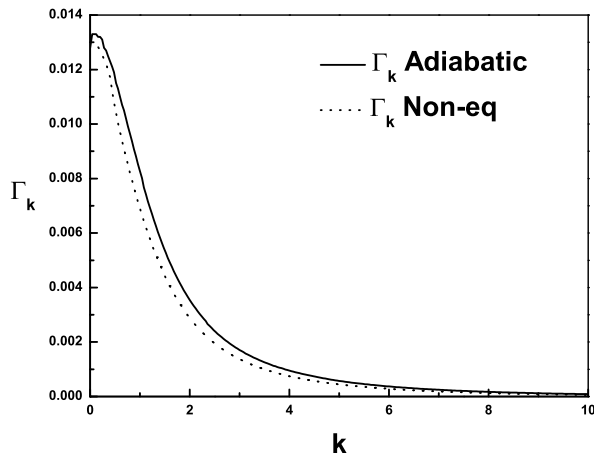$$\ddot\phi + U(\phi) \approx 0, \tag{62}$$

FIG. 3: Momentum distribution of Γ_k at $t \approx 3$ for both the adiabatic and the non-equilibrium solutions.

while the adiabatic approximation to this equation is

$$\ddot{\phi} + U(\phi) + \eta(\phi)\dot{\phi} \approx 0. \qquad (63)$$

As seen from figure 2, where $\sigma \propto |\dot{\phi}|$, the velocity $\dot{\phi}$ in the adiabatic approximation quickly approaches the "terminal velocity" $\dot{\phi} \approx -U(\phi)/\eta(\phi)$ and thereafter ϕ and $\dot{\phi}$ evolve slowly, with $\ddot{\phi} \approx 0$. According to the original equation, by contrast, $\ddot{\phi}$ is still approximately equal to $-U(\phi)$, leading to the dashed curve. Clearly, in these circumstances, the original equation of motion is not well represented by the adiabatic approximation, and this is substantially independent of how well the number densities used to calculate $U(\phi)$ and $\eta(\phi)$ are thermalized.

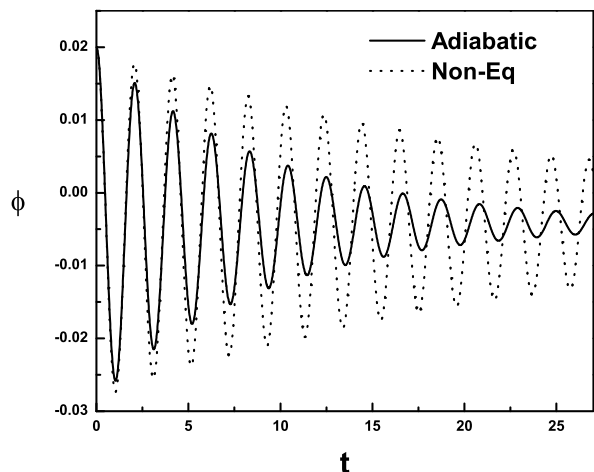An example of the more generic case of underdamped motion is shown in figure 4. We see that the nonequilibrium evo-

FIG. 4: Underdamped motion of ϕ for both the adiabatic and non-equilibrium solutions

lution shows significant damping, but that the rate of damping is overestimated by the adiabatic approximation. This differs somewhat from the case of inflaton decay into scalars [13]. In that case, the adiabatic approximation leads to a friction coef-

ficient roughly proportional to ϕ^2, and thus to an overestimate of damping at large amplitudes, but an underestimate at small amplitudes.

V. SUMMARY AND DISCUSSION

In this paper, we have sought to address a very specific issue: can the expectation value $\langle \bar{\psi}\psi \rangle$ appearing in the generic inflaton equation of motion (2), which is a non-local functional of ϕ, be adequately approximated by a local function of $\phi(t)$ and $\dot{\phi}(t)$, leading to a local equation of motion such as (1). The conclusion suggested by the results of the two preceding sections is that it cannot be, but these results are neither exact nor wholly rigorous, so it is worthwhile to summarize carefully the line of argument we have adopted.

Derivations of the local friction coefficient $\eta(\phi)$ to be found in the literature vary somewhat in the details of implementation, but generally involve the *assumption*, characteristic of linear response theory, that under suitable conditions the state of the particles can be treated according to *equilibrium* field theory. The possibility of obtaining a local equation of motion then depends on the behaviour of the relevant integral kernels, which can be investigated in perturbation theory for specific models (see, for example [10] and, for recent discussions [20, 21]). We seek to avoid this assumption by studying instead the slow-evolution limit of the *nonequilibrium* state. In principle, this is certainly the more correct procedure; in practice, however, nonequilibrium evolution is extremely hard to deal with by analytic means.

The approximation scheme sketched in sections III A and III B is the lowest non-trivial order of a partially resummed perturbation theory. It leads to a local set of evolution equations, but we find that these equations do not admit a solution in the form of a time-derivative expansion, which is needed to derive the local equation of motion (1), and consequently that this local equation of motion does not, in principle, exist. We cannot altogether rule out the possibility that this result is a feature of our approximation scheme, rather than of the exact dynamics. Quantitatively, indeed, the accuracy of the approximation is likely to be quite modest, though we have no good way of assessing this. However, the fact that it reproduces the exact dynamics of section II in the free-field limit, and that the Boltzmann-like scattering terms have a sensible kinetic-theory interpretation, offers some reassurance that the approximation captures essential features of the nonequilibrium dynamics. Moreover, any improved method of studying nonequilibrium dynamics (for example, the 2-particle-irreducible formalism developed by Berges and coworkers; see [22] and references therein, especially [23]) is intrinsically non-local. It seems very likely that any attempt to derive a local equation of motion must follow a sequence of approximations more or less equivalent to that used here, and would fail at the same point.

Even if the local equation of motion is formally not valid, it might nevertheless be a good approximation under suitable circumstances. Indeed, we can recover essentially the same result for $\eta(\phi)$ as that given by linear response methods by taking the further step of replacing nonequilibrium self-

energies with those calculated in a state of thermal equilibrium: we called this the adiabatic approximation to our non-equilibrium evolution equations. Intuitively, this would seem to be a fairly innocuous step if the evolution is fairly slow, and thermalization is not too inefficient. To test this, we obtained numerically, in section IV, solutions for the motion of ϕ both with and without the adiabatic approximation. In order to focus on the frictional effect (and to simplify the numerical calculation somewhat) we took the whole effective potential (including the local part of $\langle \bar{\psi}\psi \rangle$) to be approximated by the parabola $V = \frac{1}{2}m_\phi^2\phi^2$. While this might significantly affect the actual motion obtained for $\phi(t)$, it is irrelevant for our purpose of assessing the effect of the extra adiabatic approximation.

The results of this numerical comparison of the nonequilibrium and adiabatic approximations are broadly similar to those of the parallel calculations reported in [13] for an inflaton decaying into scalars. In both cases, we find examples of conditions for which the adiabatic approximation predicts overdamped motion (and so might seem, at least on the grounds of self-consistency, to be reliable), while the motion resulting from the "full" nonequilibrium evolution is oscillatory, with relatively little damping. (In neither case did we find conditions under which the frictional effect generated by the nonequilibrium evolution is sufficient to cause overdamping; however, we were unable to explore the large parameter space exhaustively.) Even with the adiabatic approximation, underdamped motion seems to be more typical of the model we studied, but the magnitude of the frictional effect seems to be significantly overestimated by this approximation. For the model studied in this work, indeed, it seems that friction is always overestimated by the adiabatic approximation. In the case of inflaton decay into scalars, the adiabatic approximation yields a friction coefficient that contains a factor ϕ^2, and appears to overestimate the frictional effect when ϕ is large, while underestimating it when ϕ is small.

Beyond the formal difficulty discussed in section III C, the reasons for the failure of the adiabatic approximations are hard to identify in any detail. Poor thermalization may be an issue, but as noted earlier, it seems implausible that the large discrepancies we observe are due merely to the quantitative effect of number densities that differ somewhat from a true thermal distribution. A more plausible possibility is that the discrepancies could be lessened simply by retaining more terms in the time-derivative expansion. Comparison of (62) and (63) makes it rather obvious that, if the adiabatic motion is friction limited, so that $\dot{\phi} \approx -U(\phi)/\eta(\phi)$ and $\ddot{\phi} \approx 0$, and if the same value of ϕ and the same number densities are used to evaluate both the adiabatic and the "full" equation of motion, a large discrepancy is more or less inevitable. It could be, however, that this friction-limited motion does not lead to negligible values for higher-order terms in the time-derivative expansion, and that this accounts for the discrepancy. Unfortunately, pursuing the adiabatic approximation to higher orders is rather complicated (and not entirely unambiguous, in view of the evolving temperature we employed); and to our knowledge, approximation schemes of this kind are not commonly used in the cosmological literature.

The possibility of describing the dissipation of inflaton energy by a local friction term is primarily of interest in connection with warm inflation scenarios, and it should be emphasized that the results reported here and in [13] do not in themselves determine whether or not such scenarios may be viable. To the extent that any generic result can be extracted, it seems that the adiabatic approximation overestimates the effects of dissipation: in particular, overdamped motion (and hence warm inflation) is apparently harder to achieve than the adiabatic approximation suggests. However, this may be true only of the simple models we have investigated. The somewhat discouraging conclusion is that no simple analytical approximation known to us represents dissipation reliably, and numerical investigations of the kind reported here offer a very inefficient tool for inflationary model building.

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